

Domain wall entropy of the bimodal two-dimensional Ising spin glass

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We report calculations of the domain wall entropy for the bimodal two-dimensional Ising spin glass in the critical ground state. The $L \times L$ system sizes are large with L up to 256. We find that it is possible to fit the variance of the domain wall entropy to a power function of L . However, the quality of the data distributions are unsatisfactory with large $L > 96$. Consequently, it is not possible to reliably determine the fractal dimension of the domain walls.

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I. INTRODUCTION

The short-range Ising spin glass is still a source of controversy in spite of its comparative simplicity. In brief, the exact mechanisms by which widely separated spins influence each other are not clearly understood. The system Hamiltonian, due to Edwards and Anderson¹, is

$$H = - \sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \quad (1)$$

where the nearest-neighbor exchange interactions J_{ij} are quenched random variables which, for present purposes, take random sign with equal probability. The two particular models of disorder widely studied are the bimodal, or $\pm J$, model where the interactions have fixed magnitude and the Gaussian model where they are taken from a continuous normal distribution of zero mean. In two dimensions it is generally accepted that the spin glass only exists at zero temperature^{2,3}, that is the critical temperature $T_c = 0$.

Spin correlations in two dimensions for bimodal disorder are expected to decay algebraically according to

$$\langle \sigma_0 \sigma_R \rangle^2_{av} \sim R^{-\eta} \quad (2)$$

in the ground state. To date, the best estimates from Monte Carlo simulation⁴ and exact calculation⁵ seem to agree that the exponent is $\eta = 0.14$. For Gaussian disorder, the exponent is zero since the ground state, apart from a global inversion, is unique. In contrast, bimodal disorder comes with a large ground state degeneracy corresponding to an entropy of $0.07k$ per spin^{6,7,8,9,10,11}. In spite of this it has been argued¹² that $\eta = 0$ in the thermodynamic limit. Very recently, Hartmann¹³ has estimated $\eta = 0.22$ from scaling arguments. This is in fair agreement with a number of previous estimates^{7,9,14,15,16}. Nevertheless, the important issue here is universality; that is whether or not η is positive.

Clear evidence of the importance of long range influence in the case of bimodal disorder is apparent in the current controversy regarding the lowest excited state. This issue dates back to the work of Wang and Swendsen⁷ which proposed that the energy gap should be $2J$, not

$4J$, in the thermodynamic limit. Although some work^{4,9} has supported the simple naive $4J$ scenario, it is now becoming clearer that strong disorder is indeed producing a different result for the infinite system. The energy gap has been often reported^{11,17,18} as $2J$. Further, it has even been suggested, based on a finite size scaling study of the correlation length and the spin glass susceptibility¹², that the specific heat should obey a power law. This may indicate a feature universal with Gaussian disorder. A good review of the issues involved here is given in Ref. 19. Essentially it appears to be the case that there exist low energy excitations with very long range influence.

A standard technique commonly used to investigate the long range response of a spin glass is to introduce a domain wall defect^{14,20,21}. This is done, in two dimensions, by drawing a line across the system and reversing the signs of all bonds cut by the line. With Gaussian disorder, the ground and low energy excited states are unique. Consequently, the term domain wall has a clear physical meaning and appears as an optimum fractal path that corresponds to the lowest excited state. The case of bimodal disorder is quite different as a result of the large degeneracy of both the ground and excited states. The degeneracy of the ground state is of the order of $\exp(0.07L^2)$ for a $L \times L$ lattice and that of the first excited state is probably much larger still¹¹. In particular, it is often the case that the defect system is not an excited state of the reference system. Thus, we do not have real domain walls in the same sense as with the case of Gaussian disorder. Nevertheless, we might expect some useful knowledge to be derived from a study of the bimodal system in response to reversing the signs of bonds along a line as with the Gaussian system.

Thinking of the thermodynamic limit, we can discuss this issue in the light of droplet theory²². For the case of a continuous defect distribution, Gaussian for instance, there is a unique ground state and we define a droplet as a region bounded by a closed path (or surface in three dimensions). Below the transition temperature, which is positive in three dimensions at least²³, the scaling properties of droplets of various sizes can be investigated by reversing all spins inside the closed path. This creates an excited state of the reference system and it is known that there exist low energy droplet excitations of large spatial extent. The idea here is that a domain wall de-

fect is much the same as a large droplet excitation in the thermodynamic limit. For bimodal disorder this comparison is less clear due to the huge ground state degeneracy. Certain droplets may not represent excitations at all.

The most important prediction of droplet theory is that the energy difference E_{dw} between the two systems, with and without the domain wall, can be fitted to a power function of the system size L . We have, for the spin glass,

$$\langle |E_{dw}| \rangle \sim L^\theta \quad (3)$$

where θ is known as the spin-glass stiffness exponent. In three dimensions, this exponent is found to be positive^{24,25}, at about 0.2 for both bimodal and Gaussian disorder, showing consistency with the existence of a stable spin glass phase at finite temperature. As a matter of fact the droplet theory²² was originally developed for this type of case where both the critical temperature and stiffness exponent are positive. It is also probable that the Gaussian and bimodal models fit into the same universality class²³ with respect to their transitions at finite critical temperature. Degeneracy is not an issue due to the thermal fluctuations.

In two dimensions it was not immediately obvious that droplet theory²² is entirely appropriate since the critical temperature is zero and the stiffness exponent is not positive. For Gaussian disorder the stiffness exponent is negative²⁶, $\theta \approx -0.28$, clearly indicating that the spin glass is unstable at any finite temperature. It is also remarkable that the values of θ obtained from domain wall and droplet calculations are in very good agreement. For bimodal disorder it is generally accepted that the stiffness exponent due to domain wall defects is zero^{2,27,28}. Nevertheless the situation for droplets is not clear. Hartmann¹³ has estimated the droplet stiffness exponent by constructing ground and first excited states of three models of bimodal disorder and found good agreement with Gaussian disorder. Nevertheless, universality is not shown since the correlation function exponent η is not zero.

For the case of Gaussian disorder, in two dimensions, it has been suggested^{29,30} that the domain walls are stochastic Loewner evolution processes³¹. This theory is able to relate the domain wall fractal dimension d_f to the stiffness exponent via

$$d_f = 1 + \frac{3}{4(3 + \theta)} \quad (4)$$

and the result $d_f \approx 1.27$ agrees well with the literature²⁶. There is also an interesting conjecture due to Fisch³² that $\theta = (\sqrt{6} - 3)/2$ exactly. Nevertheless, there is no good reason to believe that Eq. (4) can be used with bimodal disorder, possibly due to the degeneracy of the ground state.

The domain wall entropy S_{dw} is defined in the same manner as E_{dw} . Droplet theory²² predicts that S_{dw}

should take values with random sign and large variance. In particular, the variance is predicted to scale as

$$\langle S_{dw}^2 \rangle - \langle S_{dw} \rangle^2 \sim L^{d_f} \quad (5)$$

which provides a possible means to estimate d_f . In particular, we can use this to test the appropriateness of droplet theory for the case where both the critical temperature and the stiffness exponent tend to zero from above. This is precisely the situation we have for the two-dimensional model with bimodal disorder.

Previous estimations of the fractal dimension from direct studies of the domain wall entropy have been published. Saul and Kardar⁹ predict $d_f \approx 1.0$. Fisch³³ argues that d_f might be an increasing function of $|E_{dw}|$. The possibility that $d_f = 1.25$ in agreement with Eq. (4) is not ruled out. Finally, Lukic *et al*³⁴ have reported $d_f = 1.03(2)$. These values should also be compared with those from topological analysis of the ground state. Romá *et al*³⁵ report $d_f = 1.30(1)$ while Melchert and Hartmann³⁶ find an interval $1.095(1) \leq d_f \leq 1.395(1)$.

In this article we report calculations of the domain wall entropy on sample sizes that are much larger than anything done before. Furthermore, our method is applied at an arbitrarily small temperature and there is no need to extrapolate to the ground state. Our main conclusion is that the domain wall fractal dimension for bimodal disorder, as predicted by droplet theory, is not a well defined quantity. The reason for this is that there exists no clear prescription for its estimation if the domain wall entropy distributions are significantly far from normal. A brief overview of the method is given in Sec. II. This is followed by our results in Sec. III and a brief discussion in Sec. IV.

II. BACKGROUND

The planar Ising model is known to be isomorphic to a system of noninteracting fermions. One particular illustration³⁷ has been adapted by Blackman³⁸ for disordered systems. For the square lattice, each site is decorated with four fermions. Equivalently, we can decorate each bond with two fermions. For a system of N lattice sites, we have $4N$ fermions in total. It is useful to think of the two fermions decorating a bond to be placed one on either side. In this way a plaquette (square) is decorated with four fermions; left, right, top and bottom.

The partition function for the Ising model on a square lattice with any set of exchange interactions takes the form

$$Z = 2^N \left[\prod_{\langle ij \rangle} \cosh(J_{ij}/kT) \right] (\det D)^{1/2} \quad (6)$$

where the product is over all nearest neighbor bonds J_{ij} on the N site lattice and D is a skew-symmetric ($4N \times$

$4N$) matrix. The square root of the determinant of D is also called the Pfaffian. Essentially, it represents the sum over all closed lattice polygons and is equal to the product of all the positive eigenvalues of D .

The calculation of the partition function with bimodal disorder has been described in much detail previously⁸ and a simple summary should suffice here. At zero temperature, D is a singular matrix with a set of degenerate zero eigenvalues exactly equal in number to the total number of frustrated plaquettes. In order to extract the physics of the system, degenerate state perturbation theory is applied at an arbitrarily low temperature. The defect eigenvalues occur in pairs and approach zero as some power of $\exp(-2J/kT)$

$$\epsilon = \pm \frac{1}{2} X \exp(-2Jr/kT) \quad (7)$$

where r is an integer (an order of perturbation theory) and X is a real number that is independent of temperature and depends only on the configuration of frustrated plaquettes. The ground state energy and entropy can be expressed exactly as

$$E = -2NJ + 2J \sum_d r_d \quad (8)$$

$$S = k \sum_d \ln X_d \quad (9)$$

where the sums are over all defect eigenstate pairs.

To summarise the perturbation theory, we first write the matrix D exactly as the sum of two terms $D = D_0 + \delta D_1$, where $\delta = 1 - t$ with $t = \tanh(J/kT)$. Of course $t = 1$ and $\delta = 0$ in the ground state. Both of the matrices D_0 and D_1 are independent of temperature. The matrix D_0 has eigenvectors localised inside each frustrated plaquette; expanded in the basis of the four decorating fermions. It is these localised states that form the defect basis for the perturbation theory. The matrix D_1 is 2×2 block diagonal in the pairs of fermions decorating the bonds (one fermion either side). All degeneracy at first order is lifted by diagonalizing D_1 in the defect basis. For example, we can think of just two neighboring frustrated plaquettes. The perturbation theory gives one defect pair with $r_d = 1$ and $X_d = 1$.

In general, the first order calculation will leave some zero eigenvalues. The corresponding eigenvectors of D_1 form the basis for second order. We can imagine a system of two next nearest neighbor frustrated plaquettes. Clearly $r_d = 2$ and $X_d = 1$ or 2 depending on the arrangement. In order to show this we need to use the continuum Green's function^{5,8} $g_c = g_{c1} + g_{c2}$. The matrix g_{c1} is 4×4 block diagonal in the four fermions inside each plaquette and clearly allows us to connect two frustrated plaquettes, across an unfrustrated plaquette. The second order calculation is performed by diagonalizing

$D_2 = D_1 g_{c1} D_1$. The matrix g_{c2} is, just like D_1 , 2×2 block diagonal in the pairs of fermions decorating bonds. A proof that g_{c2} is irrelevant for the ground state has been given in Ref. 5.

For higher orders we require Green's functions for states whose degeneracy has already been lifted. We define, for $r \geq 1$,

$$G_r = - \sum_{i=1}^{N(r)} |r, i\rangle \langle 1/\epsilon_r^i | r, i| \quad (10)$$

where $|r, i\rangle$ denotes state i (with eigenvalue ϵ_r^i) in the set of states whose degeneracy was lifted at order r ; there are $N(r)$ of these states. At third order the matrix to be diagonalized is $D_3 = D_2(1 + G_1 D_1) g_{c1} D_1$ and, generally at arbitrary order $D_n = D_{n-1}(1 + G_{n-2} D_{n-2}) \cdots (1 + G_1 D_1) g_{c1} D_1$. The perturbation theory is applied order by order until all degeneracy is lifted.

The scheme outlined above allows exact calculations of energy and entropy in the ground state. The method is fully gauge invariant in that it depends only on the number and distribution of frustrated plaquettes. Furthermore, there is no requirement to extrapolate to the ground state. The Pfaffian is not calculated at any particular numerical value of the temperature. We believe that this method is the best available for calculating the ground state entropy of large lattices, although matching algorithms are better for the energy².

We have used periodic boundary conditions in one dimension. The cylindrically wound frustrated patch was nested in an unfrustrated system of infinite extent in the second dimension. In this scheme, the introduction of a domain wall defect is particularly simple. The two plaquettes at the ends of the defect, one on each side of the patch, change their status; frustrated to unfrustrated or vice versa. For a perfect ferromagnet this gives a domain wall energy proportional to L as required. For a fully frustrated system the defect would make no real difference since the domain wall energy would be independent of L . For the spin glass the domain wall energies are all multiples of $4J$ since we have L even. The probability that a plaquette is frustrated is expected to be close to 0.5 and it is conceivable that the system with the domain wall could be interchanged with its reference system in another realization of disorder. This is consistent with the prediction of droplet theory that the domain wall entropy has random sign.

Since the domain wall entropy is generally a small difference between two larger numbers, we have taken great care with the floating point computations. Although our method is analytically exact, it is subject to numerical propagation error on the computer. Ill-conditioned disorder realizations were detected by calculating the correlation function along a path around the cylinder and repeated in arbitrary precision arithmetic as necessary.

III. RESULTS

We have calculated the domain wall entropy for bimodal disorder on $L \times L$ lattices where $L = 8, 12, 16, 24, 32, 48, 64, 96, 128, 192$ and 256 . For sizes $L \leq 128$, 10^5 random samples were taken. We also took 4×10^4 for $L = 192$ and 10^4 for $L = 256$.

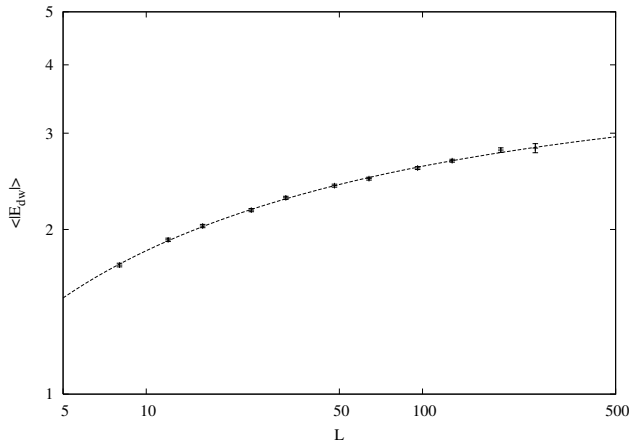


FIG. 1: The domain wall energy (in units of J) as a function of system size L . The error bars are two standard deviations of the mean and the curve is a best fit to the form $A - BL^{-p}$ following Ref. 2.

To establish the credentials of our boundary scheme, we have calculated the domain wall energy. The data is shown in Fig. 1 where the error bars are two standard deviations of the mean. Following Ref. 2, we have fitted the data to a function $A - BL^{-p}$ and find $A = 3.7(2)$, $B = 3.2(1)$ and $p = 0.23(4)$. We note that the fit is approaching saturation from below. We believe that this is as a consequence of our boundary conditions. The probability of a zero energy ($E_{dw} = 0$) domain wall is found to decrease with L , contrary to the situation with free boundary conditions in the unwound dimension³⁹. Furthermore, since we only use even values of L , the defect energies are all multiples of $4J$. The quality of the non-linear fit⁴⁰ is $Q = 0.91$. Attempts to fit a power law were not successful. For instance a fit for $L \geq 96$ completely missed the point at $L = 256$. We conclude that our method is reliable although larger system sizes $L > 256$ are required to conclude more convincingly that $\theta = 0$.

The variance of the domain wall entropy S_{dw} is shown in Fig. 2. A power law fit for $24 \leq L \leq 96$ predicts that, according to Eq. (5), the fractal dimension of the domain walls is $d_f = 1.090(8)$ where the quality of the fit is $Q = 0.42$. A similar fit ($Q = 0.30$) of $\langle |S_{dw}|^2 \rangle$ gives $1.080(9)$ which agrees well, indicating good quality data distributions.

A second power law fit for $L \geq 96$, also shown in Fig. 2, reveals a significantly higher value $d_f = 1.30(3)$ with $Q = 0.16$. However, a fit of $\langle |S_{dw}|^2 \rangle$ gives only $1.23(2)$. Although the quality $Q = 0.05$ is lower, the difference is too large to be disregarded. The reason for

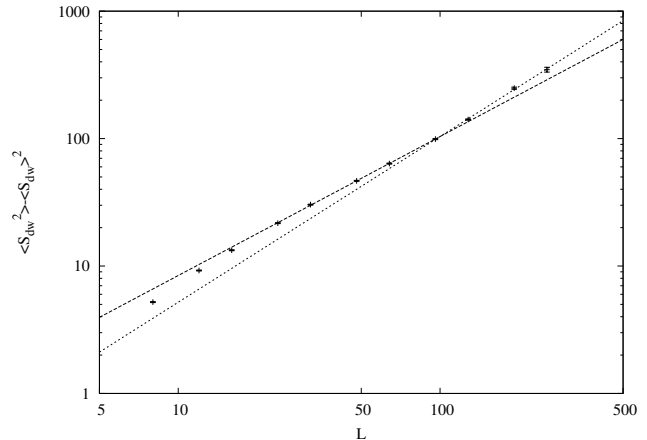


FIG. 2: The variance of the domain wall entropy as a function of system size L . The error bars are two standard deviations of the mean. Two power law fits are shown; for $24 \leq L \leq 96$ (dashed line) and $L \geq 96$ (dotted line). The powers are respectively $1.090(8)$ and $1.30(3)$.

this discrepancy must lie in the quality of the distributions for S_{dw} . For $L = 256$, for example, the distribution has skewness 0.64 and kurtosis 2.06 . Although the mean $\langle S_{dw} \rangle = 3.53$ is still much less than the variance (≈ 350) it reflects a significant sign disparity. The bias most likely arises due to correlations in the distribution of frustrated plaquettes. Incidentally, the corresponding distributions for E_{dw} are of excellent quality. For $L = 256$, the skewness and kurtosis are respectively -0.006 and 0.07 .

IV. DISCUSSION

In summary, the distributions of the domain wall entropy for large L , with bimodal disorder in two dimensions, are found to deviate significantly from normal. In consequence, even if we assume that droplet theory²² is appropriate, it is unable to prescribe exactly how to get the domain wall fractal dimension. This does not necessarily mean that droplet theory is entirely wrong. It just does not give the whole story, only an approximation, for this system; having a large ground state degeneracy, a zero critical temperature and a zero stiffness exponent. Of course, it is quite probable that corrections to scaling are large and difficult to manage. This is actually a rather likely scenario in view of the poor results for fitting the ground state energy with cylindrical winding in one dimension²⁸. Scaling corrections are an issue probably related to strong correlations in the distribution of frustrated plaquettes for large L . All gauge invariant quantities like entropy depend only on the frustrated plaquette distribution; nothing else. The prediction of droplet theory that the domain wall entropy is normally distributed with zero, or very small, mean and large variance probably relates to an assumption that a defect system occurs

as a reference system in another realization of disorder. This assumption may not be true if the frustrated plaquette distributions are strongly correlated, as is likely in view of the anomalous behaviour of the degeneracy of the first excitations mentioned earlier. A further scenario is that due to Hartmann¹³, which proposes that droplet theory is actually inappropriate for estimating the stiffness exponent of domain wall defects. If this is correct, it is also unlikely that the fractal dimension of domain walls has anything to do with droplet theory. Nevertheless, our results do indicate an approximate appropriateness for droplet theory in the sense that all the possible estimates for d_f do not seem unreasonable.

Previous studies of the domain wall entropy have worked with much smaller system sizes. Saul and Kardar⁹ had sizes up to $L = 36$ and found $d_f \approx 1.0$, while Lukic *et al*³⁴ used sizes up to $L = 50$ and fitted $\langle |S_{dw}| \rangle$ to find $d_f = 1.03(2)$. Fisch³³ used sizes up to $L = 48$ and has introduced the idea that the domain wall entropy may be significantly correlated with energy. It is argued that an effective d_f increases as a function of $|E_{dw}|$ and convergence to the value 1.25 consistent with Eq. (4) is not ruled out. We have tested these predictions and find that, for $E_{dw} = 0$, the response is in fact much stronger in both the intermediate and large size regimes. For $L \geq 96$ we find $d_f = 1.43(4)$ while, for $E_{dw} \neq 0$, $d_f = 1.22(3)$. Also, the probability of finding a disorder realization with $E_{dw} = 0$ is under 0.5 for $L > 24$,

much less than 0.75. The cause of these discrepancies is most probably due to boundary conditions and system size. We do not have any evidence from our work that the particular values of E_{dw} are significant for the droplet theory.

We also note that a topological analysis of ground states³⁵ predicts $d_f = 1.30(1)$. Essentially, the technique measured the average length of domain walls. These lengths respond faster than just L since the domain walls bend to avoid the rigid lattice. Nevertheless, only one ground state configuration was studied for each disorder realization, completely ignoring the entropy issue. A study in a similar vein³⁶ looks at the properties of minimum energy domain walls and places the fractal dimension in an interval $1.095(1) \leq d_f \leq 1.395(1)$. This may agree to some extent with the point that it is not possible, or very difficult, to actually pin down the value of d_f .

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